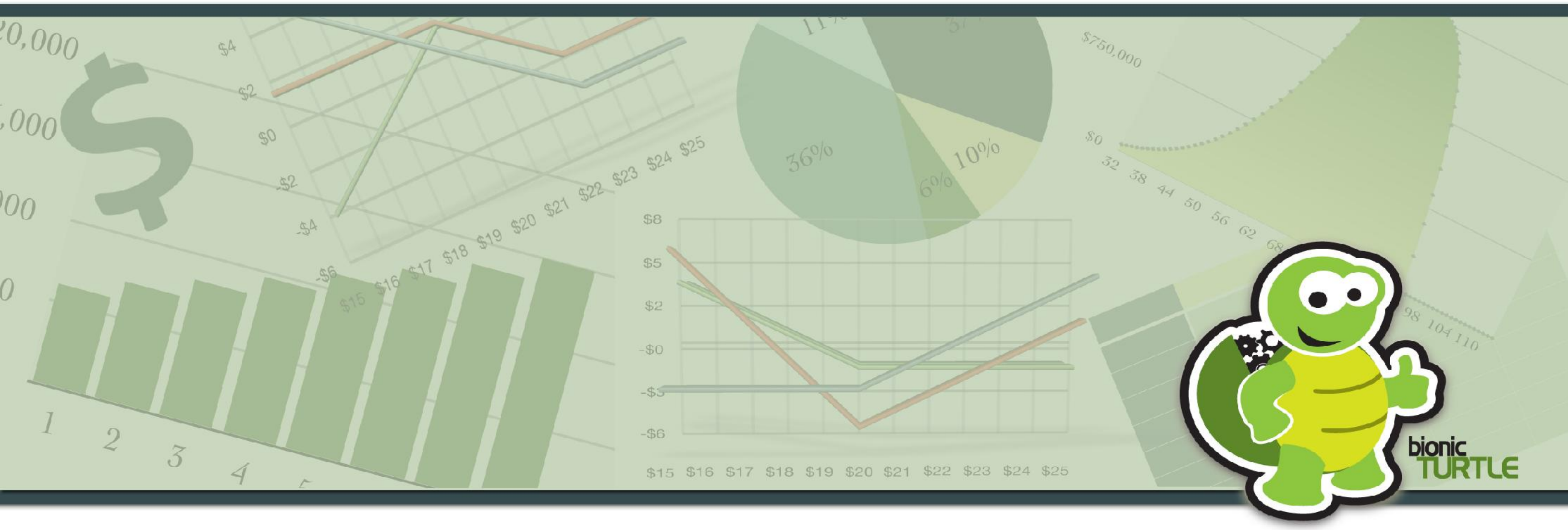


P1.T2. Quantitative Analysis

Chapter 10: Stationary Time Series

Bionic Turtle FRM Study Notes

By David Harper, CFA FRM CIPM
www.bionicturtle.com

Chapter 10: Stationary Time Series

Chapter 10: Stationary Time Series

- **Describe the requirements for a series to be covariance stationary.**
- **Define the autocovariance function and the autocorrelation function.**
- **Define white noise, describe independent white noise and normal (Gaussian) white noise.**
- **Define and describe the properties of autoregressive (AR) processes.**
- **Define and describe the properties of moving average (MA) processes.**
- **Explain how a lag operator works.**
- **Explain mean reversion and calculate a mean-reverting level.**
- **Define and describe the properties of autoregressive moving average (ARMA) processes.**
- **Describe the application of AR, MA, and ARMA processes.**
- **Describe sample autocorrelation and partial autocorrelation.**
- **Describe the Box-Pierce Q-statistic and the Ljung-Box Q statistic.**
- **Explain how forecasts are generated from ARMA models.**
- **Describe the role of mean reversion in long-horizon forecasts.**
- **Explain how seasonality is modeled in a covariance-stationary ARMA.**

Selected key concepts:

- Key concepts to be included in next revision

Describe the requirements for a series to be covariance stationary.

Consider a time series, $\{\ldots y_{-2}, y_{-1}, y_0, y_1, \ldots y_t \ldots\}$ ordered in a sequence that contains observations for each time period.

The series does not need be ordered in time. A series can even be formed by recording, for instance, residential property prices from one location to another location thus forming a *spatial* series comprising sample residential prices within a distance of 35 miles. Theoretically, a time series is estimated to range between infinite point of time in past and future, which is why, such range is difficult to interpret. Hence, a limited subset of this infinite series called Sample Path $\{y_0, y_1, \ldots, y_t\}$ is used for practical purpose.

For a series to remain **covariance stationary**, the following three conditions are required:

1. **Stable Mean:** The mean of the series should be stable (and therefore constant) over the observation window. Assuming mean of the series, $y_0, y_1, \ldots y_t$, at time t is:

$$\mathbf{E(y_t) = \mu_t}$$

The series will have stable mean only if the mean μ(t) remains constant for all the values of t, $\mu \neq f(t)$; i.e., the mean of the series should not be a function of time. Therefore, if the mean is stable, we can drop the (t) subscript from the above equation:

$$\mathbf{E(y_t) = \mu}$$

2. **Stable Covariance:** Estimating covariance stability is a delicate yet vital process and is carried out with the help of **Autocovariance Function**. The autocovariance $\boldsymbol{\gamma(t,\tau)}$ for time displacement τ is the covariance between γ(t) and γ(t-τ) and is given by:

$$\boldsymbol{\gamma}(\mathbf{t},\boldsymbol{\tau}) = \mathbf{cov}(\mathbf{y_t}, \mathbf{y_{t-\tau}}) = \mathbf{E}(\mathbf{y_t} - \boldsymbol{\mu})(\mathbf{y_{t-\tau}} - \boldsymbol{\mu})$$

Also, considering the prerequisite for covariance stationarity that the covariance structure should be stable during the observation period then the autocovariance must depend upon tau, τ, only and not on time, t. Hence irrespective of all the values of t, the autocovariance can be rewritten as:

$$\boldsymbol{\gamma}(\mathbf{t},\boldsymbol{\tau}) = \boldsymbol{\gamma}(\boldsymbol{\tau})$$

In other words, the autocovariance is exclusively dependent upon displacement factor tau, $\boldsymbol{\tau}$, and not on t. Furthermore, autocovariance is a symmetric function i.e. for all $\boldsymbol{\tau}$:

$$\boldsymbol{\gamma}(\boldsymbol{\tau}) = \boldsymbol{\gamma}(-\boldsymbol{\tau})$$

Therefore, the signage of the tau, $\boldsymbol{\tau}$, value does not have any impact on the autocovariance function and the function is only dependent upon displacement. Furthermore when $\boldsymbol{\tau} = 0$, i.e. in case of 0 displacement, the autocovariance becomes equivalent to the variance of the series itself.

$$\boldsymbol{\gamma}(\mathbf{0}) = \mathbf{cov}(\mathbf{y_t}, \mathbf{y_t}) = \mathbf{var}(\mathbf{y_t})$$

3. **Finite Autocovariance** (i.e., third condition of covariance stationary)**:** The last condition for a series to be covariance stationary is that the resulting autocovariance at zero displacement i.e. $\gamma(0)$ should be finite. It can be proven statistically that autocovariance at T = 0 is comparatively the highest in absolute terms. Therefore, if $\gamma(0)$ is finite i.e. $\gamma(0) < \infty$ then the all the other resulting autocovariance values should also be finite.

Covariance Stationarity is also called second-order stationarity or weak stationarity because despite stringent requirements for mean and variance of a series, it does not dictate any prerequisites pertaining to skewness and kurtosis of the series.

Define the autocovariance function and the autocorrelation function.

Autocovariance Function

The autocovariance function is the covariance between two values of a series, separated by a time lag called displacement, here denoted with Greek tau, τ. Thus, autocovariance, $\gamma(t, \tau)$ between y(t) and y(t – τ) is given by:

$$\gamma(t,\tau) = \text{cov}(y_t, y_{t-\tau}) = E(y_t - \mu)(y_{t-\tau} - \mu)$$

- Autocovariance provides a snapshot of the cyclical dynamics exhibited by covariance stationary series and is one of the major components in assessing if a series fulfills the condition of covariance stationarity.
- Autocovariance is exclusively dependent upon displacement factor, τ and when derived against varying (rising) values of τ, it graphically represents the impact of displacement on autocovariance and subsequently on the covariance stationarity of the series.

Autocorrelation

Correlation coefficient between two random variables x and y is given by the classic formula:

$$\text{corr}(x,y) = \frac{\text{cov}(x,y)}{\sigma_x \sigma_y}$$

The covariance between the two variables is scaled by the product of the standard deviations of each variable. This division is also called *standardizing* because the resultant correlation is unitless; aka, scale-free.

- Although both covariance and correlation define the degree of association between two random variables, correlation is *easier to interpret and comprehend*
- Covariance has an unlimited range (e.g., it could be in the millions or very small numbers), but correlation must lie between -1.0 to +1.0, inclusive: for instance, a covariance result ranging in millions may not give a precise answer whereas; a correlation coefficient of 0.94 clearly portrays extremely close association between two variables.

For the same reason that correlation is preferred to covariance, the autocorrelation function is *generally preferred* to autocovariance.

The autocorrelation function, ρ(τ), is given by dividing the autocovariance function by the variance, i.e.

$$\boldsymbol{\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)}, \tau = 0, 1, 2, \ldots}$$

Autocorrelation is derived as a cross-correlation between a series with itself with a time lag, specifically displacement, τ. In this way, autocorrelation reveals the association between the observations of a series recorded with a time lag.

If we assume covariance stationarity, then the variance of γ(t) and γ(t-τ) are constant at γ(0). Therefore, the autocorrelation, ρ(τ), between γ(t) and γ(t-τ) is given by:

$$\boldsymbol{\rho(\tau) = \frac{cov(y_t, y_{t-\tau})}{\sqrt{var(y_t)}\sqrt{var(y_{t-\tau})}} = \frac{\gamma(\tau)}{\sqrt{\gamma(0)}\sqrt{\gamma(0)}} = \frac{\gamma(\tau)}{\gamma(0)}}$$

At a (time) lag of zero displacement (i.e., τ = 0) the autocorrelation will be equivalent to 1.0 because the series will be perfect correlated with itself:

$$\boldsymbol{\rho(0) = \frac{\gamma(0)}{\gamma(0)} = 1}$$

Autocorrelation with zero lag serves no purpose and is completely uninformative. However, **autocorrelation with non-zero displacement is used extensively** for assessing the (cyclical) dynamics of a series.

Partial Autocorrelation

Partial Autocorrelation, p(τ), is useful in assessing the ***exclusive*** **correlation between a variable and its lagged value**, (y_t and y_{t-T}), while removing any linear association with {y_{t-1}, y_{t-2} … y_{t-T-1}}.

Like the autocorrelation function, at displacement of zero (0), partial correlation also equals 1.0 and is uninteresting at zero displacement.

We typically plot both the autocorrelation and partial autocorrelation functions. The graph should begin at displacement 1.0 to give meaningful results.

Graphical Interpretation of Autocorrelation[1]

The autocorrelation function for a covariance stationary series should be decaying with the increase in displacement and approaches zero (0) for larger displacement. The autocorrelation graph shown below in *Figure 1* displays non-decaying autocorrelation with increasing displacement, so it illustrates an autocorrelation function that does not belong to a covariance stationary time series.

For a covariance stationary series, the autocorrelation must be decaying with increasing displacement and the decay rate is exclusively dependent upon the series.

- Figure 1 does not dampen and it not stationary
- Figure 2 displays gradual one-sided dampening
- Figure 3 displays an abrupt drop to zero autocorrelation at displacement = 15
- Figure 4 shows a viable autocorrelation falling below zero and move back and forth before finally dropping to zero at a larger displacement value

Figure 1: Autocorrelation function, non-dampening (not stationary)

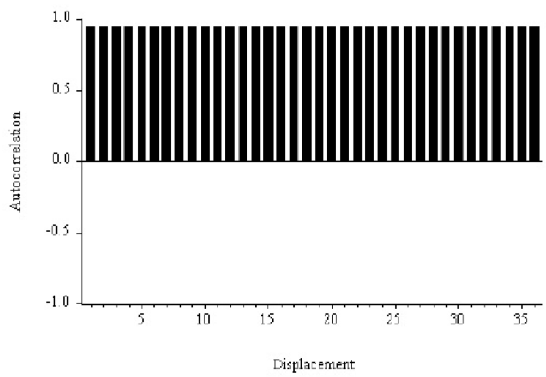


Figure 2: Autocorrelation function, one-sided gradual dampening

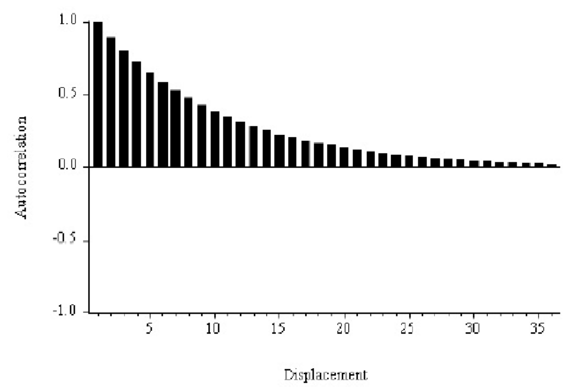


Figure 3: Autocorrelation function, sharp cutoff

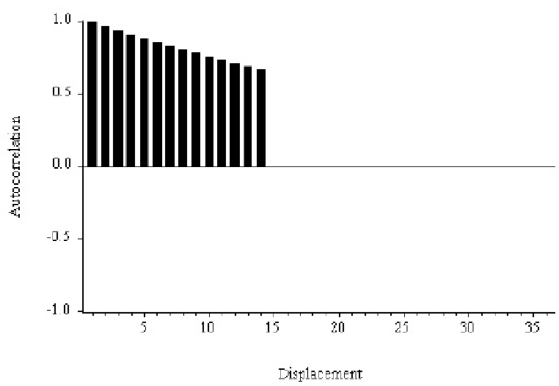
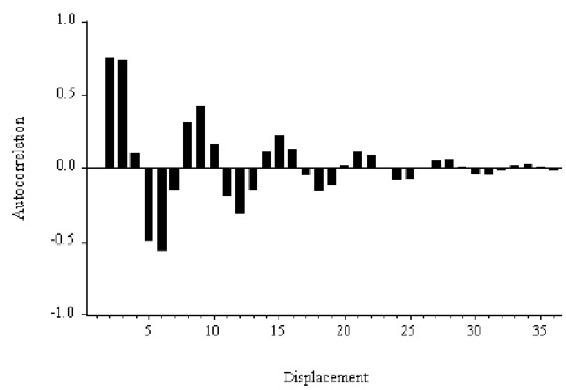


Figure 4: Autocorrelation function: gradual damped oscillation



[1] Diebold, Francis, Elements of Forecasting