P1.T4. Valuation & Risk Models

Chapter 12. Applying Duration, Convexity, and DV01

Bionic Turtle FRM Study Notes

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- **Describe a one-factor interest rate model and identify common examples of interest rate factors.**
- **Define and compute the DV01 of a fixed income security given a change in yield and the resulting change in price.**
- **Calculate the face amount of bonds required to hedge an option position given the DV01 of each.**
- **Define, compute, and interpret the effective duration of a fixed income security given a change in yield and the resulting change in price.**
- **Compare and contrast DV01 and effective duration as measures of price sensitivity.**
- **Define, compute, and interpret the convexity of a fixed income security given a change in yield and the resulting change in price.**
- **Explain the process of calculating the effective duration and convexity of a portfolio of fixed income security.**
- **Describe an example of hedging based on effective duration and convexity.**
- **Construct a barbell portfolio to match the cost and duration of a given bullet investment, and explain the advantages and disadvantages of bullet versus barbell portfolios**

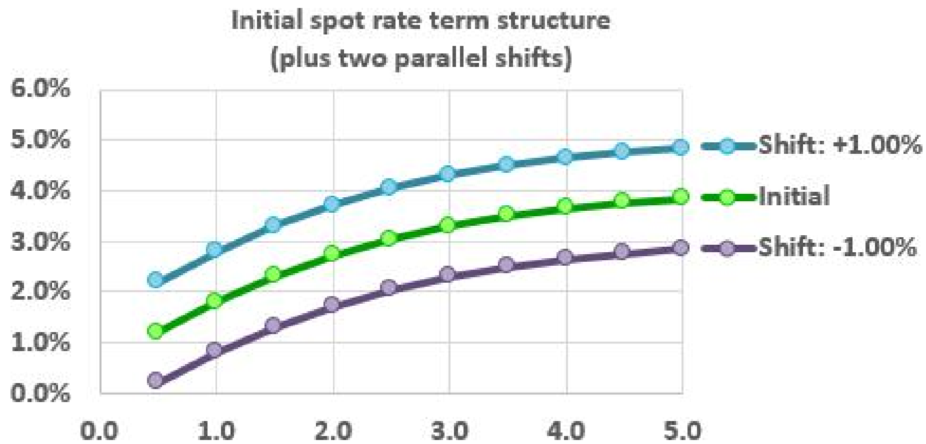
Selected key concepts:

- Interest rate factors include spot, forward, par rate, and yield-to-maturity (aka, yield). Along with the instantaneous short spot rate (popular in the dynamic term structure models like Vasicek), **yield** is the most common single-factor interest rate. A single-factor interest rate model describes *term structure movements* with a single factor. While not all single-factor models imply parallel shifts, a yield shock does imply an approximately parallel shift in the spot rate term structure.
- The **DV01** is a bond's dollar change in value given a one basis point decline in the yield.
- **Macaulay duration** is the bond's weighted average maturity where the weight is the present value of each cash flow as a fraction of the bond's price. The Macaulay duration of a zero-coupon bond is the bond's maturity. **Modified duration** (as a percentage) is the approximate percentage change in price associated with a 1.0% (100 basis points) change in yield; e.g., we expect a bond with a modified duration of 8.5 years to experience a price change of 8.5% if the yield changes by 1.0%. Modified duration = Macaulay duration ÷ (yield / m) where m is the number of periods for the year; therefore, under continuous compounding modified duration and Macaulay duration are identical.
- **Effective duration** is a numerical method that approximates modified duration. While approximate, it is robust to non-vanilla bonds; e.g., embedded options. Effective duration is given by $\frac{\mathbf{Price}_{y0-\Delta y}-\mathbf{Price}_{y0+\Delta y}}{2\times\Delta\mathbf{y}\times\mathbf{Price}_{y0}}$ and it approximates modified duration $\left(\cong -\frac{1}{\mathrm{P}}\frac{\partial \mathrm{P}}{\partial y}\right)$
- **Effective convexity** is given by $\frac{\mathbf{P}_{+}+\mathbf{P}_{-}-2\times\mathbf{P}_{0}}{\mathbf{P}_{0}(\Delta\mathbf{y})^2}$ and it approximates convexity $\left(\cong \frac{1}{\mathrm{P}}\frac{d^2P}{dy^2}\right)$

Describe a one-factor interest rate model and identify common examples of interest rate factors.

Yield (YTM) is the most common single-factor interest rate

To illustrate several ideas, we will evaluate the price/yield of **one bond** under three different upward-sloping spot rate term structures (see the plot to the right): the **"Initial" (in green)** and two parallel shifts of 100 basis points, an **upward shift,** and a **downward shift**.



Our single bond pays a 4.0% semi-annual coupon. Below are the price/yield values under each term structure scenario (**Initial Term, upward shift,** and **downward shift**). In the upward shift, +1.0% is added to each spot rate. By design, therefore, we are modeling a *parallel shift*. The theoretical prices are present values (aka, discounted cash flows) and these prices inform the yields. When the curve shifts up by 100 basis points, the yield increases by 99.85 bps (0.998%); when the curve shifts down by 100 bps, the yield decreases by 99.85bps. Durations (Macaulay ~ modified) and convexity are calculated; they are not our current focus but are included so we can see how they react to yield shifts.

Face $100.00 | Coupon 4.00%

	Cash Flows	Intial Term		Shift up		Shift down	
Term (t)	(FV)	Spot	PV CF	1.00%	PV CF	-1.00%	PV CF
0.5	$2.00	1.20%	$1.99	2.20%	$1.98	0.20%	$2.00
1.0	$2.00	1.80%	$1.96	2.80%	$1.95	0.80%	$1.98
1.5	$2.00	2.30%	$1.93	3.30%	$1.90	1.30%	$1.96
2.0	$2.00	2.71%	$1.90	3.71%	$1.86	1.71%	$1.93
2.5	$2.00	3.04%	$1.85	4.04%	$1.81	2.04%	$1.90
3.0	$2.00	3.30%	$1.81	4.30%	$1.76	2.30%	$1.87
3.5	$2.00	3.50%	$1.77	4.50%	$1.71	2.50%	$1.83
4.0	$2.00	3.65%	$1.73	4.65%	$1.66	2.65%	$1.80
4.5	$2.00	3.76%	$1.69	4.76%	$1.62	2.76%	$1.77
5.0	$102.00	3.84%	$84.33	4.84%	$80.31	2.84%	$88.59
	Price:		$100.976		$96.557		$105.632
	Yield:		3.784%		4.782%		2.785%
	Yield Δ				0.9985%		-0.9985%
	Duration, Mac ~ mod		4.58 ~ 4.49		4.56 ~ 4.46		4.59 ~ 4.53
	Convexity		21.31		21.04		21.59

Let's use the simulation on the previous page to make several observations:

- An *interest rate* is a generic variable. Specific interest rates include **spot rates** (aka, zero rates), **forward rates**, **par rates** (aka, par yields), and **yields** (aka, yield-to-maturity). The learning objective asks us to identify common examples of interest rate factors. The most common interest rate factor is **a short-term spot rate**. In the FRM's Part 2, we will study Term Structure models (e.g., Ho-Lee, Vasicek). Most of those Term Structure models assume the interest rate factor is an *instantaneous* short-term spot rate. When the *dynamic changes to a term structure*, it can be described by a single factor (such as the instantaneous spot rate), then we have a single-factor model. There is a subtle distinction between a *factor* and an *interest rate*. A factor is a general variable or "container;" it is not necessarily an interest rate.
- The **term structure** typically refers to the *spot (aka, zero) rate term structure*[1]. The simulation above illustrates three different (and parallel) spot rate term structures. We did not illustrate a "yield curve" because the yield (YTM) is a single value that summarizes (as complex average of) the vector of spot rates. The yield must plot a flat (horizontal) line[2]. There is not really a term structure of yields (YTMs): yield is a single interest rate that is variant to each bond's cash flows and price. To describe a term structure, we require a vector (The obvious *exception* is the flat term structure famously used in exams. A flat term structure is super convenient because all of these interest rates—spot, forward, par, and yield, are the same when the term structure is flat!).
- When we shifted the spot rate term structure, above, in parallel by 100 basis points, the yields shifted by approximately, *but not exactly,* 100 basis points (0.9985%)! Because the yield is a single value (that is *variant to* the bond's cash flows and therefore having no objective existence independent of the bond[3]), it does **not** make sense to simulate a "parallel shift in the yield curve." Rather, we can shock the (single) yield value. Our simulation models a parallel shift in the spot rate term structure with a single factor model (e.g., plus 100 basis points added to all spot rates on the term structure), and this shift itself *very closely approximates* the resultant change in yield.
- In practice—by using yield-based duration, convexity, and DV01—we are shocking the yield and this shock implies (by way of necessary approximation) a parallel shift in the underlying term structure. The yield is a single-factor interest rate. Single-factor models do not necessarily imply a parallel shift[4], but the yield-to-maturity (YTM) does. Therefore, our yield-based DV01, duration, and convexity metrics also *implicitly assume (by way of very close approximation) a parallel shift* in the term structure.
- If we want to be sophisticated, we can model any non-parallel term structure shift by directly editing the vector of spot rates. This is very multi-factor! More likely, we'd shock a few selected key rates (e.g., key rate shift) which are also multi-factor. The simplest approach is to shock the yield and implicitly assume a parallel shift in the term structure.

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[1] We can refer to the forward rate term structure but in the previous chapter we saw that spot rates and forward rates are mathematically redundant: a given spot rate term structure implies a forward rate term structure, and vice-versa

[2] Although yield-to-maturity plots a horizontal line, as a single factor, please note that **par yields** are different: there is a par yield term structure and the multi-factor key rate shift technique prefers the par yield term structure. "Yield" connotes yield-to-maturity, it does not connote par yield. If we want to refer to par yields, we need to be explicit.

[3] Consider that, given a defined spot rate term structure, the yield changes whenever the price changes,

[4] The Vasicek model (https://en.wikipedia.org/wiki/Vasicek_model) is a single-factor model that is not parallel shift.

Define and compute the DV01 of a fixed income security given a change in yield and the resulting change in price.

The dollar value of an 0.01% (DV01) is the bond's dollar change in value given a one-basis point ***decline*** in the yield; the negative sign ("-") enforces a positive DV01 if the bond's price increases when the yield drops per the most natural dynamic:

$$\mathbf{DV_{01}} = -\frac{\Delta P}{10{,}000 \times \Delta y}$$

Importantly, the DV01 is related to modified duration (D). Modified duration is the bond's *percentage change* in price given a unit change in the yield:

$$D \equiv -\frac{\Delta P}{P \times \Delta y} \therefore \mathbf{DV_{01}} = \frac{D \times \mathbf{P}}{10{,}000} \rightarrow \mathbf{D} = \frac{\mathbf{DV_{01}}}{\mathbf{P}}(10{,}000)$$

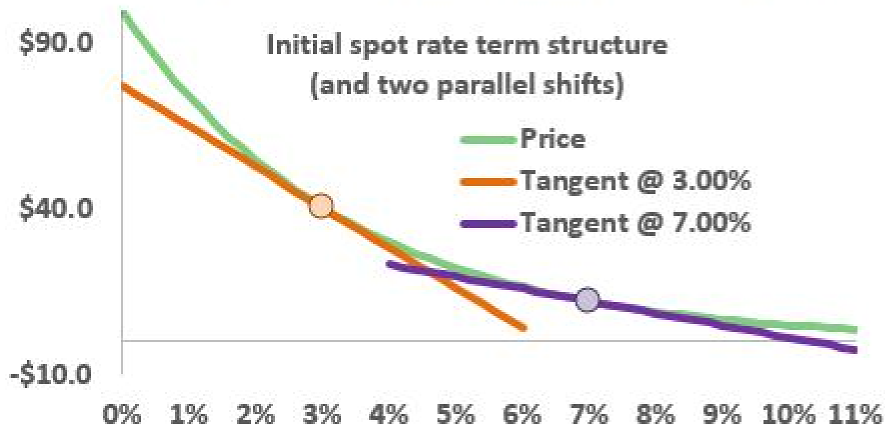
… which produces one of the most useful relationships in this chapter:

$$\mathbf{DV_{01}} = \frac{D_{modified} \times Price}{10{,}000} = \frac{dollar\ duration}{10{,}000}$$

For example: DV01. The following example (see exhibit and graph on the next page) assumes a zero-coupon bond with 30.0 years to maturity. The DV01 is computed for five different continuous yields (1.0%, 3.0%, 5.0%, 7.0% and 9.0%). For convenience, in this example, the yields are continuous (not discrete). The reason is that, if the yield is given with continuous compounding, the durations (Macaulay and modified) are identical and it is trivial to compute the slope of the tangent lines illustrated below. Continuous compounding is academic (not realistic but enables simpler calculations. The purpose here is to illustrate the relationship of DV01 to the other metrics; understanding the relationships is the key to avoiding the confusion that commonly occurs when studying duration and its sibling terms. The next example illustrates DV01 when the yields are given semi-annual compounding; the next example is more realistic for an exam.

- Because this is a zero-coupon, its Macaulay duration is equal to its maturity of 30.0 years. As we shall see later, the modified duration is given by the Macaulay duration divided by (1 + yield/k) where (k) is the number of periods per year; e.g., k equals two under semi-annual compound frequency. Under continuous compounding, $k \rightarrow \infty$ such the divisor is equal to 1.0. In this way, zero-coupon bonds where the yield is continuous are convenient because their durations both match the bond's maturity. Consequently, this bond's modified duration is also 30.0 years (see second row).

Face	$100		Maturity (yrs)	30.0	
Yield (continuous)	1.00%	3.00%	5.00%	7.00%	9.00%
Price, P	$74.082	$40.657	$22.313	$12.246	$6.721
Duration (Mac = mod), D	30.0	30.0	30.0	30.0	30.0
Tangent Slope = -D × P	-$2,222	-$1,220	-$669	-$367	-$202
DV01 = (-D × P)/10,000	**$0.2222**	**$0.1220**	**$0.0669**	**$0.0367**	**$0.0202**
Less one bps (-0.010%)	0.99%	2.99%	4.99%	6.99%	8.99%
New Price	$74.304	$40.779	$22.380	$12.282	$6.741
DV01	**$0.2226**	**$0.1222**	**$0.0670**	**$0.0368**	**$0.0202**



For example: DV01 (continued)

- At each yield (1.0%, 3.0%, 5.0%, 7.0%, 9.0%), we show the tangent line's slope, which is the first derivative, ∂P/ ∂y, and is equal to (-duration × price); e.g., at 3.0% yield, the slope is -30 × 40.657 = -1,200. The negative of this slope is the **dollar duration (DD)**; e.g., at 3.0% yield, the dollar duration is 1,220; DD is the product of price and duration.
- If we divide the dollar duration by 10,000, we retrieve the DV01. At 3.0%, given the dollar duration is 1,220, the DV01 is $0.1220. This is an analytical approach. But more typically, we compute the DV01 by "effectively" shocking the yield by one basis point …
- At 3.0%, we can see that if we subtract one basis point, the revised yield is 2.99%. At this yield, the bond's price is $100.00 * exp(-0.02990 * 30) = $40.779. Hence the DV01 is equal to $40.779 - $40.657 = $0.1220. In this way ("effectively" which means numerically, as opposed to analytically), we observe that the DV01 is $0.1220 because that is the increase in the price if we drop the yield by one basis point.
- We've also observed that the DV01 is simply a re-scaled dollar duration: it is 1/10,000th of the dollar duration[5], which itself is the negative of the tangent.

[5] The dollar duration is (the negative of) the tangent's slope. As a pure first derivative, it is the price change per one unit (1.0 = 100%) of yield which is 10,000 basis points: 100 bps (per 1.0%) × 100 percentage points = 10,000. The DV01 re-scales this slope to give us the dollar change per 1/10,000th full 100% of yield, which is the price change per one basis point. In the above example, you will notice slight differences in the two DV01s at 5.0% and 7.0% yields; e.g., $0.0669 versus $0.0670. The first DV01 ($0.0669 at 3.0%) is the re-scaled first derivative such that it linearly approximates the exact (but non-linear) second DV01 ($0.0670 at 3.0%).