P1.T4. Valuation & Risk Models

Chapter 9. Pricing Conventions, Discounting, and Arbitrage

Bionic Turtle FRM Study Notes

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- **Define discount factor and use a discount function to compute present and future values.**
- **Define the "law of one price", explain it using an arbitrage argument, and describe how it can be applied to bond pricing.**
- **Identify arbitrage opportunities for fixed income securities with certain cash flows.**
- **Identify the components of a U.S. Treasury coupon bond, and compare and contrast the structure to Treasury STRIPS, including the difference between P-STRIPS and C-STRIPS.**
- **Construct a replicating portfolio using multiple fixed-income securities to match the cash flows of a single given fixed income security.**
- **Differentiate between "clean" and "dirty" bond pricing and explain the implications of accrued interest with respect to bond pricing.**
- **Describe the common day-count conventions used in bond pricing.**

Selected key concepts:

- A discount factor, d(t) or df(t), is the present value of one dollar ($1.00) received at future time (t). Each future maturity has a different discount factor, and we expect them to decrease as maturity increases because the same dollar is worth less further into the future. A set of discount factors is a *discount function* and it maps to a specific spot rate (aka, zero rate) term structure. But discount factors are better than spot rates because they already incorporate the compound frequency information.
- The *law of one price* says that each maturity on the term structure has one and only one discount factor. If a bond's price deviates sufficiently from these discount factors, then arbitrage trades should enforce their approximate accuracy.
- If a bond's observed market (aka, traded) price is higher than its model (aka, theoretical per DCF) price, the bond is "trading rich." If a bond's observed market (aka, trade) price is lower than its model (aka, theoretical per DCF) price, the bond is "trading cheap." The law of one price cannot let the prices diverge too much, but we do expect bonds to trade cheap/rich due to frictions and technical factors, especially liquidity and supply/demand.
- The bond's flat price equals its full price minus accrued interest:

  PV = Full Price (P) = Flat price (p) + Accrued interest (AI). Equivalently,
  Flat price (p) = Full Price (P) – Accrued interest (A).

  The **Full price** is also called Cash, Invoice or Dirty Price.
  The **Flat price** is also called Quoted or Clean Price
- The accrued interest (AI) is based on the *day count convention*: Actual/Actual (Treasury bonds), Actual/360 (money market), or 30/360 (e.g., US corporate and municipal bonds).
- STRIPS are Treasury bonds that have been "stripped" into their zero-coupon components, either: C-STRIP coupons (fungible) or P-STRIPS principle (not fungible and therefore inherit rich/cheap property of their origin bond).

Define discount factor and use a discount function to compute present and future values.

The discount factor, d(t) or df(t), is the **present value of one dollar ($1.00) to be received at future time (t)**. The *discount function* is the set of discount factors. Consider the hypothetical spot rate term structure below and the corresponding discount factors. Please note that we have assumed semi-annual compound frequency to match the maturity intervals:

Maturity (t)	0.5	1.0	1.5	2.0
Spot rate	0.40%	1.20%	2.00%	3.00%
Discount function	0.998004	0.988107	**0.970590**	**0.942184**
PV of $1.00 rec'd (t)	$99.8004	$98.8107	**$97.0590**	$94.2184
FV of $1.00	$1.00200	$1.01204	**$1.03030**	$1.06136

- If the eighteen-month discount factor, d(1.5) = 0.970590, then the present value of $1.00 to be received in 1.5 years is 97.0590 cents (see **orange cells above**).
- Similarly, for the same discount factor, $1.00 received today has a future value of $1.030301 in eighteen months (1.5 years):

$$\frac{\$1}{d(1.5)} = \frac{\$1}{0.970590} = \$1.030301$$

- *How did we calculate the discount factors from the spot rate?* Per the definition, we simply discounted one dollar to its present value: df(t) = $1.00/(1+z/2)^(2*t) where (z) is the spot rate. For example, **df(2.0)= (1+0.030/2)^(-2*2.0) = 0.942184; see purple cell**.

The **discount function is the series of discount factors** that correspond to a series of times to maturity (t). For example, a discount function is the series of discount factors: d (0.5), d (1.0), d (1.5), d (2.0).

A **key advantage** of a discount factor is *the discount factor already incorporates the compound frequency*: we don't need to deliberate the issue of compound frequency with discount factors (hence the saying "discount factors never lie"!). For example, consider again the eighteen-month (1.5 year) spot rate of 2.0% from the exhibit above. Given this 2.0% 1.5-year spot rate, how do we determine the price of a zero-coupon bond that matures in 1.5 years? We have at least three valid approaches depending on the compound frequency selected:

- If we discount annually, the bond's price is $100/(1+0.020)^1.5 = $97.0733
- If we discount semi-annually, the bond's price is $100/(1+0.020/2)^(2*1.5) = $97.0590
- If we discount continuously, the bond's price is $100*exp(-0.20*1.5)= $97.0446

This demonstrates how the spot rate (aka, zero rate) by itself is *incomplete* until we specify its associated compound frequency. The discount factor, however, does not suffer this ambiguity; it already incorporates the compound frequency information. We can simply multiply it by the future value to retrieve the present value. For this reason, we might consider discount factors to be superior to spot/zero rates.

For example (Tuckman's Treasury Bonds)[1]**:** The table below combines Tuckman's Tables 1.1 and 1.2 in order to retrieve discount factors from Treasury bond prices. We *do expect the discount factors to decline as the maturity increases*: as the same one dollar ($1.00) is paid further into the future, it should be worth less today.

Selected U.S. Treasury Bond Prices as of May 28, 2010					
Time	Maturity	Years	Coupon	Price	Discount Function
0.50	11/30/2010	0.5	1.250%	$100.550	0.99925
1.00	5/31/2011	1.0	4.875%	$104.513	0.99648
1.50	11/30/2011	1.5	4.500%	$105.856	0.99135
2.00	5/31/2012	2.0	4.750%	$107.966	0.98532
2.50	11/30/2012	2.5	3.375%	$105.869	0.97520
3.00	5/31/2013	3.0	3.500%	$106.760	0.96414
3.50	11/30/2013	3.5	2.000%	$101.552	0.94693
4.00	5/31/2014	4.0	2.250%	$101.936	0.93172
4.50	11/30/2014	4.5	2.125%	$100.834	0.91584

- To retrieve the discount function, we should start with the first discount factor because this is a bond with only a single cash flow. As the first row shows, this $1\,{}^{1}/_{4}$ bond (i.e., coupon rate is 1.250% per annum but pays semi-annually) is due in six months (T = 0.5) and has a market price of $100.55. We can extract the discount factor, d (0.5) for November 30, 2010, by equating the price of the bond to the present value of its future cash flows, namely its principal plus coupon payment, all times the discount factor for funds to be received in six months.

  $$\$\mathbf{100.55} = \boldsymbol{d}(\mathbf{0.5})[\$\mathbf{100} + (\mathbf{1.25}\%/\mathbf{2})] \rightarrow \boldsymbol{d}(\mathbf{0.5}) = \mathbf{0.99925}$$

- Now that we know the d(0.5) discount factor, we can solve for the one-year discount factor, d(1.0). The price of this bond should equal the present value of its cash flows:

  $$\$\mathbf{104.513} = [\boldsymbol{d}(\mathbf{0.5}) \times (\mathbf{4.875}\%/\mathbf{2})] + \boldsymbol{d}(\mathbf{1})\,[\$\mathbf{100} + (\mathbf{4.875}\%/\mathbf{2})]$$

  Given the solution for $d(0.5)$, this equation can be solved for $d(1) = 0.99648$.

- The third row is solved is a similar manner:

  $$\$\mathbf{105.856} = [\boldsymbol{d}(\mathbf{0.5}) \times (\mathbf{4.5}\%/\mathbf{2})] + [\boldsymbol{d}(\mathbf{1}) \times (\mathbf{4.5}\%/\mathbf{2}) + \boldsymbol{d}(\mathbf{1.5})[\$\mathbf{100} + (\mathbf{4.5}\%/\mathbf{2})]$$

  Given the solutions for $d(0.5)$ and $d(1)$, we can solve for $d(1.5) = 0.99135$.

- The remainder of the discount factors can be solved in this somewhat bootstrap-like approach. As long as we start at the first discount factor and solve for its value, each subsequent discount factor should require solving for only one unknown.

[1] Tuckman, Bruce and Angel Serrat. Fixed Income Securities: Tools for Today's Markets. Wiley 2012. Based on Tuckman's Tables 1.1 and 1.2, but spreadsheet was hand constructed by David Harper.

Treasury Bill (GARP Chapter 9.1)

Treasury bills are money market instruments; i.e., short-term debt as opposed to capital market instruments which are long-term debt. They belong in a category called *discount instruments* because they are quoted using a discount rate: their interest rate is expressed as a percentage of their final face value rather than the initial purchase price (therefore, their interest rate is not the effective interest rate).

The relationship between the quoted price, Q, and the cash price, C, is given by[2]:

$$Q = \frac{360}{n}(100 - C) \rightarrow C = 100 - \frac{Qn}{360}$$

For example, in the table below, we can infer the cash price of the T-bill that matures on June 7, 2018, and has a quoted bid price of 1.640:

$$\text{Bid cash price, C} = 99.5900 = 100 - \frac{1.640 \times 90}{360}$$

Treasury Bill Quotes on				Mar 9, 2018	
		Quote		Cash Price	
Maturity	Days	Bid	Ask	Bid	Ask
Mar 15, 2018	6	1.345	1.335	99.9776	99.9778
Mar 22, 2018	13	1.395	1.385	99.9496	99.9500
Mar 29, 2018	20	1.495	1.485	99.9169	99.9175
Apr 5, 2018	27	1.540	1.530	99.8845	99.8853
Apr 12, 2018	34	1.568	1.558	99.8519	99.8529
Apr 19, 2018	41	1.560	1.550	99.8223	99.8235
Apr 26, 2018	48	1.548	1.538	99.7936	99.7949
May 3, 2018	55	1.565	1.555	99.7609	99.7624
May 10, 2018	62	1.603	1.593	99.7239	99.7257
May 17, 2018	69	1.623	1.613	99.6889	99.6908
May 24, 2018	76	1.628	1.618	99.6563	99.6584
May 31, 2018	83	1.630	1.620	99.6242	99.6265
Jun 7, 2018	**90**	**1.640**	**1.630**	**99.5900**	**99.5925**
Jun 14, 2018	97	1.648	1.638	99.5560	99.5587
Jun 21, 2018	104	1.680	1.670	99.5147	99.5176
Jun 28, 2018	111	1.678	1.668	99.4826	99.4857
Jul 5, 2018	118	1.720	1.710	99.4362	99.4395
Jul 12, 2018	125	1.730	1.720	99.3993	99.4028
Jul 19, 2018	132	1.740	1.730	99.3620	99.3657
Jul 26, 2018	139	1.775	1.765	99.3147	99.3185
Aug 2, 2018	146	1.803	1.793	99.2688	99.2728

[2] 2020 Financial Risk Management Part I: Valuation and Risk Models, 10th edition. Pearson Learning Solutions, 10/2019. However, their original source is Chapter 6 of the John Hull's Options, Futures, and Other Derivatives (OFOD), 10th edition, Pearson Education